

2.8 Integration

Standard forms^a

$$\int u \, dv = [uv] - \int v \, du \quad (2.353) \quad \int uv \, dx = v \int u \, dx - \int \left(\int u \, dx \right) \frac{dv}{dx} \, dx \quad (2.354)$$

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} \quad (n \neq -1) \quad (2.355) \quad \int \frac{1}{x} \, dx = \ln|x| \quad (2.356)$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax} \quad (2.357) \quad \int x e^{ax} \, dx = e^{ax} \left(\frac{x}{a} - \frac{1}{a^2} \right) \quad (2.358)$$

$$\int \ln ax \, dx = x(\ln ax - 1) \quad (2.359) \quad \int \frac{f'(x)}{f(x)} \, dx = \ln f(x) \quad (2.360)$$

$$\int x \ln ax \, dx = \frac{x^2}{2} \left(\ln ax - \frac{1}{2} \right) \quad (2.361) \quad \int b^{ax} \, dx = \frac{b^{ax}}{a \ln b} \quad (b > 0) \quad (2.362)$$

$$\int \frac{1}{a+bx} \, dx = \frac{1}{b} \ln(a+bx) \quad (2.363) \quad \int \frac{1}{x(a+bx)} \, dx = -\frac{1}{a} \ln \frac{a+bx}{x} \quad (2.364)$$

$$\int \frac{1}{(a+bx)^2} \, dx = \frac{-1}{b(a+bx)} \quad (2.365) \quad \int \frac{1}{a^2+b^2x^2} \, dx = \frac{1}{ab} \arctan \left(\frac{bx}{a} \right) \quad (2.366)$$

$$\int \frac{1}{x(x^n+a)} \, dx = \frac{1}{an} \ln \left| \frac{x^n}{x^n+a} \right| \quad (2.367) \quad \int \frac{1}{x^2-a^2} \, dx = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| \quad (2.368)$$

$$\int \frac{x}{x^2 \pm a^2} \, dx = \frac{1}{2} \ln|x^2 \pm a^2| \quad (2.369) \quad \int \frac{x}{(x^2 \pm a^2)^n} \, dx = \frac{-1}{2(n-1)(x^2 \pm a^2)^{n-1}} \quad (2.370)$$

$$\int \frac{1}{(a^2-x^2)^{1/2}} \, dx = \arcsin \left(\frac{x}{a} \right) \quad (2.371) \quad \int \frac{1}{(x^2 \pm a^2)^{1/2}} \, dx = \ln|x + (x^2 \pm a^2)^{1/2}| \quad (2.372)$$

$$\int \frac{x}{(x^2 \pm a^2)^{1/2}} \, dx = (x^2 \pm a^2)^{1/2} \quad (2.373) \quad \int \frac{1}{x(x^2-a^2)^{1/2}} \, dx = \frac{1}{a} \operatorname{arcsec} \left(\frac{x}{a} \right) \quad (2.374)$$

^a a and b are non-zero constants.



Trigonometric and hyperbolic integrals

$\int \sin x \, dx = -\cos x$	(2.375)	$\int \sinh x \, dx = \cosh x$	(2.376)
$\int \cos x \, dx = \sin x$	(2.377)	$\int \cosh x \, dx = \sinh x$	(2.378)
$\int \tan x \, dx = -\ln \cos x $	(2.379)	$\int \tanh x \, dx = \ln(\cosh x)$	(2.380)
$\int \csc x \, dx = \ln \left \tan \frac{x}{2} \right $	(2.381)	$\int \operatorname{csch} x \, dx = \ln \left \tanh \frac{x}{2} \right $	(2.382)
$\int \sec x \, dx = \ln \sec x + \tan x $	(2.383)	$\int \operatorname{sech} x \, dx = 2 \arctan(e^x)$	(2.384)
$\int \cot x \, dx = \ln \sin x $	(2.385)	$\int \coth x \, dx = \ln \sinh x $	(2.386)
$\int \sin mx \cdot \sin nx \, dx = \frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)} \quad (m^2 \neq n^2) \quad (2.387)$			
$\int \sin mx \cdot \cos nx \, dx = -\frac{\cos(m-n)x}{2(m-n)} - \frac{\cos(m+n)x}{2(m+n)} \quad (m^2 \neq n^2) \quad (2.388)$			
$\int \cos mx \cdot \cos nx \, dx = \frac{\sin(m-n)x}{2(m-n)} + \frac{\sin(m+n)x}{2(m+n)} \quad (m^2 \neq n^2) \quad (2.389)$			

Named integrals

Error function	$\operatorname{erf}(x) = \frac{2}{\pi^{1/2}} \int_0^x \exp(-t^2) \, dt$	(2.390)
Complementary error function	$\operatorname{erfc}(x) = 1 - \operatorname{erf}(x) = \frac{2}{\pi^{1/2}} \int_x^\infty \exp(-t^2) \, dt$	(2.391)
Fresnel integrals ^a	$C(x) = \int_0^x \cos \frac{\pi t^2}{2} \, dt; \quad S(x) = \int_0^x \sin \frac{\pi t^2}{2} \, dt$	(2.392)
	$C(x) + i S(x) = \frac{1+i}{2} \operatorname{erf} \left[\frac{\pi^{1/2}}{2} (1-i)x \right]$	(2.393)
Exponential integral	$\operatorname{Ei}(x) = \int_{-\infty}^x \frac{e^t}{t} \, dt \quad (x > 0)$	(2.394)
Gamma function	$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, dt \quad (x > 0)$	(2.395)
Elliptic integrals (trigonometric form)	$F(\phi, k) = \int_0^\phi \frac{1}{(1-k^2 \sin^2 \theta)^{1/2}} \, d\theta \quad (\text{first kind})$	(2.396)
	$E(\phi, k) = \int_0^\phi (1-k^2 \sin^2 \theta)^{1/2} \, d\theta \quad (\text{second kind})$	(2.397)

^aSee also page 167.



Definite integrals

$$\int_0^{\infty} e^{-ax^2} dx = \frac{1}{2} \left(\frac{\pi}{a} \right)^{1/2} \quad (a > 0) \quad (2.398)$$

$$\int_0^{\infty} x e^{-ax^2} dx = \frac{1}{2a} \quad (a > 0) \quad (2.399)$$

$$\int_0^{\infty} x^n e^{-ax} dx = \frac{n!}{a^{n+1}} \quad (a > 0; n = 0, 1, 2, \dots) \quad (2.400)$$

$$\int_{-\infty}^{\infty} \exp(2bx - ax^2) dx = \left(\frac{\pi}{a} \right)^{1/2} \exp\left(\frac{b^2}{a}\right) \quad (a > 0) \quad (2.401)$$

$$\int_0^{\infty} x^n e^{-ax^2} dx = \begin{cases} 1 \cdot 3 \cdot 5 \cdot \dots \cdot (n-1)(2a)^{-(n+1)/2} (\pi/2)^{1/2} & n > 0 \text{ and even} \\ 2 \cdot 4 \cdot 6 \cdot \dots \cdot (n-1)(2a)^{-(n+1)/2} & n > 1 \text{ and odd} \end{cases} \quad (2.402)$$

$$\int_0^1 x^p (1-x)^q dx = \frac{p!q!}{(p+q+1)!} \quad (p, q \text{ integers} > 0) \quad (2.403)$$

$$\int_0^{\infty} \cos(ax^2) dx = \int_0^{\infty} \sin(ax^2) dx = \frac{1}{2} \left(\frac{\pi}{2a} \right)^{1/2} \quad (a > 0) \quad (2.404)$$

$$\int_0^{\infty} \frac{\sin x}{x} dx = \int_0^{\infty} \frac{\sin^2 x}{x^2} dx = \frac{\pi}{2} \quad (2.405)$$

$$\int_0^{\infty} \frac{1}{(1+x)x^a} dx = \frac{\pi}{\sin a\pi} \quad (0 < a < 1) \quad (2.406)$$

